ON THE TEMPERATURE DISTRIBUTION OF A VISCOUS IN-COMPRESSIBLE FLUID IN A CIRCULAR PIPE UNDER UNSTEADY RATE OF HEAT ADDITION

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ABSTRACT. In the present paper expressions for the temperature distributions in a circular pipe are derived when viscous incompressible fluid is flowing through it. The term of dissipation due to friction is not neglected and the rate of heat addition (*i*) varies linearly with time, and (*ii*) decreases exponentially with time. The difference in the temperature distribution introduced by the inclusion of the dissipiation term has also been discussed.

INTRODUCTION

This paper consists of two parts. In Part A the temperature distribution in a circular pipe when viscous incompressible fluid is flowing through it with the rate of heat addition varying linearly with time is discussed. An expression for the temperature distribution is obtained in dimensionless form. This consists of two parts, the one varies linearly with dimensionless time Fourier modulus $T_1 = \frac{k't}{R^2}$ (R radius) and the other is transient part of temperature, which vanishes in the limit as t tends to infinity. It is also seen that the contribution of the transient part is insignificant when $T_1 > 1$.

In part B the temperature distribution in the same pipe is studied when viscous incompressible fluid is flowing through it with the rate of heat addition decreasing exponentially with time. An expression for the temperature has been obtained taking

$$\frac{1}{\rho c_v} \frac{\partial Q}{\partial t} = \sum_{n=1}^{\infty} a_n e^{-nt},$$

which has been compared with that of Lal's result (1964) where he has obtained the expression by neglecting the dissipation term. Our expression contains some additional terms and the reason for this has been discussed. The result is in complete agreement with similar results obtained by Ballabh (1959) and Snnedon (1951) where Ballabh has obtained the expression for the velocity by using the method of superposability and Snnedon has discussed the heat flow under exponentially decreasing temperature gradient.

Here the expressions for the temperature distributions in both the parts are derived with the conditions that the surface $r = R$ (i) has zero initial temperature and (ii) is always being kept at zero temperature.

1. The energy equation (Pai, 1956) in the present case is

$$\frac{\partial T}{\partial t} = \frac{1}{\rho c_v} \frac{\partial Q}{\partial t} + k' \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + cr^2, \qquad \ldots \quad (1.1)$$

where $k' = \frac{k}{\rho c_v}$ and $c = \frac{4\mu u_m^2}{\rho c_v R^4}$ are constants, and u_m represents the maximum velocity in the pipe. The last term in the equation (1.1) is dissipation due to friction and is not neglected in the present investigation.

PART A

2. Rate of heat addition varies linearly with time.
We, therefore, assume that

$$\frac{1}{\rho c_v} \frac{\partial Q}{\partial t} = at. \qquad (2.1)$$

Equation (1.1) then becomes

$$\frac{\partial T}{\partial t} \quad at + k' \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + cr^2. \qquad (2.2)$$

Now let $\bar{T} = \int_0^\infty e^{-st}.T\, dt$ be the Laplace transform of T and let T_0 be the initial value of T.

Multiplying equation (2.2) by e^{-st} and integrating between the limits 0 to ∞, we get

$$\frac{\partial^2 \bar{T}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{T}}{\partial r} - \frac{s}{k'} \bar{T} = -\frac{1}{k'} \left(T_0 + \frac{cr^2}{s} + \frac{a}{s^2} \right). \qquad \ldots \quad (2.3)$$

Equation (2.3) can be solved by the method of variation of parameter and we get the solution as

$$\bar{T} = AI_0(rp) + BK_0(rp) + \phi(r, s), \qquad \ldots \quad (2.4)$$

where $$p = \sqrt{\frac{s}{k'}}, \quad \text{and}$$

$$k' \phi(r, s) = K_0(rp) \int \left(T_0 + \frac{cr^2}{s} + \frac{a}{s^2} \right) . I_0(rp).rdr$$

$$-I_0(rp) \int \left(T_0 + \frac{cr^2}{s} + \frac{a}{s^2} \right) .K_0(rp)rdr.$$

Now we shall find T_0.

Initially the rate of heat addition is zero and the temperature is steady in the pipe.

Hence $\dfrac{\partial T_0}{\partial t}=0$ and we obtain $$\frac{d^2T_0}{dr^2}+\frac{1}{r}\frac{dT_0}{dr}=-\frac{c}{k'}r^2 \tag{2.5}$$

The boundary conditions are

$$T_0=\text{finite} \quad \text{when } r=0$$

and $$T_0=0 \text{ when } r=R.$$

The solution of equation (2.5) under these boundary conditions is

$$T_0=\frac{c}{16k'}(R^4-r^4).$$

Substituting this value of T_0 in the expression for $\phi(r,s)$ and using certain recurrence relations of Bessel functions, we obtain

$$\phi(r,s)=\frac{c}{16k'}\left(\frac{R^4-r^4}{s}\right)+\frac{a}{s^3}.$$

Then

$$\bar{T}=AI_0(rp)+BK_0(rp)+\frac{C}{16k'}\left(\frac{R^4-r^4}{s}\right)+\frac{a}{s^3}. \tag{2.6}$$

The boundary conditions for $\bar{T}$ are

$$\bar{T}=\text{finite when } r=0$$

and $$\bar{T}=0 \quad \text{when} \quad r=R.$$

Applying these boundary conditions, we get

$$\bar{T}=\frac{C}{16k'}\left(\frac{R^4-r^4}{s}\right)+\frac{a}{s^3}-\frac{a}{s^3}\cdot\frac{J_0(irp)}{J_0(iRp)}.$$

Now applying Laplace inversion theorem, we obtain

$$T=\frac{C}{16k'}(R^4-r^4)+\frac{1}{4k'}(R^2-r^2)\,at-\frac{a}{64k'^2}(3R^2-r^2)(R^2-r^2)$$

$$+2a\sum_{m=1}^{\infty}\frac{R^4}{k'^2\alpha_m^5}\cdot\frac{J_0\left(\frac{r\alpha_m}{R}\right)}{J_1(\alpha_m)}\cdot e^{-\frac{k'\alpha_m^2t}{R^2}} \tag{2.7}$$

where α_m are the positive roots of $J_0(\alpha) = 0$

At time $t = 0$, $T = \frac{C}{16k'}(R^4 - r^4)$. Hence from equation (2.7) by putting

$$t = 0, \quad \text{we get} \quad \sum_{m=1}^{\infty} \frac{J_0\left(\frac{r\alpha_m}{R}\right)}{\alpha_m^5 J_1(\alpha_m)} = \frac{1}{128}\left(3 - \frac{r^2}{R^2}\right)\left(1 - \frac{r^2}{R^2}\right).$$

Writing $r/R = \eta$ so that η is less than 1, we get

$$\sum_{=1}^{\infty} \frac{J_0(\eta\alpha_m)}{\alpha_m^5 J_1(\alpha_m)} = \frac{1}{128}(3-\eta^2)(1-\eta^2).$$

Putting $\eta = 0$, we get

$$\sum_{m=1}^{\infty} \frac{}{\alpha_m^5 J_1(\alpha_m)} = \frac{3}{128}.$$

Now we make equation (2.7) dimensionless by introducing

$$\tau = \frac{T}{\theta}, \quad \frac{r}{R} = \eta, \quad T_1 = \frac{k't}{R^2},$$

where θ is a characteristic temperature.

We then get

$$\tau = b_0(1-\eta^4) + bT_1(1-\eta^2) - \frac{b}{16}(3-\eta^2)(1-\eta^2)$$

$$+8b \sum_{m=1}^{\infty} \frac{1}{\alpha_m^5} \cdot \frac{J_0(\eta\alpha_m)}{J_1(\alpha_m)} \cdot e^{-\alpha_m^2 T_1}, \quad \ldots \quad (2.8)$$

where $b_0 = \frac{cR^4}{16k'\theta}$ and $b = \frac{aR^4}{4k'^2\theta}$ are clearly dimensionless numbers.

We now take

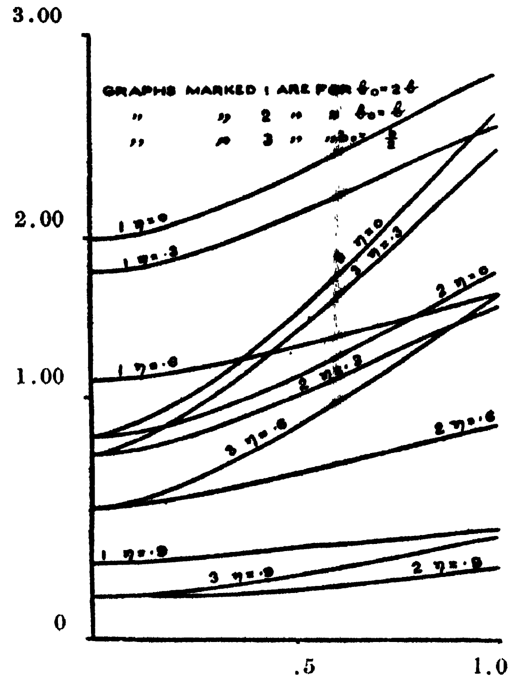
$$\tau = \tau_1 + \tau_2, \quad \text{where}$$

$$\tau_1 = b_0(1-\eta^4) + b\,T_1(1-\eta^2) - \frac{b}{16}(3-\eta^2)(1-\eta^2),$$

and

$$\tau_2 = 8b \sum_{m=1}^{\infty} \frac{1}{\alpha_m^5} \cdot \frac{J_0(\eta\alpha_m)}{J_1(\alpha_m)} \cdot {}^{-\alpha_m^2 T_1}$$

The graphs for fixed $\eta(\eta = 0,\ 0.3,\ 0.6,\ 0.9)$ showing the variation of τ with the parameter T_1 (dimensionless time Fourier modulus) have been drawn in three cases $b = 1$, $b_0 = 1$; $b = 2$, $b_0 = 1$; and $b = 1$; $b_0 = 2$ in the range $T_1 = 0$ to $T_1 = 1$.



Graph showing variation of τ with T_1

The graphs beyond $T_1 = 1$ have not been drawn because τ_2 is very small compared to τ_1 when $T_1 > 1$, hence the transient part is insignificant and τ varies linearly with T_1 in this range. From the graphs it is observed that τ incrsases with T_1 for fixed η. It is also seen that for any T_1, τ decreases with the increase of η and it is maximum when $\eta = 0$. It means that points near the axis of the cylinder are having more temperature than the points which are far from the axis of the cylinder.

PART B

3. Rate of heat addition decreases exponentially with time.

We assume that
$$\frac{1}{\rho c_v} \frac{\partial Q}{\partial t} = \sum_{n=1}^{\infty} a_n e^{-nt}. \tag{3.1}$$

Equation (1.1) then becomes
$$\frac{\partial T}{\partial t} = \sum_{n=1}^{\infty} a_n e^{-nt} + k' \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + cr^2 \tag{3.2}$$

Let $\bar{T} = \int_0^\infty e^{-st} . T dt$ be the Laplace transform of T and let T_0 be the initial value of T.

Multiplying equation (3.2) by e^{-st} and integrating between the limits 0 to ∞, we get

$$\frac{\partial^2 \bar{T}}{\partial r^2}+\frac{1}{r}\frac{\partial \bar{T}}{\partial r}-\frac{s}{k'}\bar{T}=-\frac{1}{k'}\left[T_0+\frac{cr^2}{s}+\sum_{n=1}^{\infty}\frac{a_n}{(s+n)}\right].$$

The above equation can be solved by the method of variation of parameter and we get the solution as

$$\bar{T}=AI_0(rp)+BK_0(rp)+\phi(r,s), \qquad \ldots (3.3)$$

where $$p=\sqrt{\frac{s}{k'}}, \text{ and}$$

$$k'.\phi(r,s)=K_0(rp)\int\left[T_0+\frac{cr^2}{s}+\sum_{n=1}^{\infty}\frac{a_n}{(s+n)}\right]I_0(rp).rdr$$

$$-I_0(rp)\int\left[T_0+\frac{cr^2}{s}+\sum_{n=1}^{\infty}\frac{a_n}{(s+n)}\right]K_0(rp).rdr.$$

Here $T_0=\frac{c}{16k'}(R^4-r^4)$ as obtained in Part A.

Substituting this value of T_0 in the expression for $\phi(r,s)$ and using certain recurrence relations of Bessel functions, we get

$$\phi(r,s)=\frac{c}{16k'}\left(\frac{R^4-r^4}{s}\right)+\sum_{n=1}^{\infty}\frac{a_n}{s(s+n)}.$$

Then

$$\bar{T}=AI_0(rp)+BK_0(rp)+\frac{C}{16k'}\left(\frac{R^4-r^4}{s}\right)+\sum_{n=1}^{\infty}\frac{a_n}{s(s+n)}. \qquad \ldots (3.4)$$

Applying the previous boundary conditions, we get

$$\bar{T}=\frac{C}{16k'}\left(\frac{R^4-r^4}{s}\right)+\left[1-\frac{J_0(irp)}{J_0(iRp)}\right]\sum_{n=1}^{\infty}\frac{a_n}{s(s+n)}. \qquad \ldots (3.5)$$

Now applying Laplace inversion theorem, we get

$$T=\frac{c}{16k'}(R^4-r^4)-\sum_{n=1}^{\infty}\frac{a_n}{n}\left[1-\frac{J_0\left\{r\left(\frac{n}{k'}\right)^{\frac{1}{2}}\right\}}{J_0\left\{R\left(\frac{n}{k'}\right)^{\frac{1}{2}}\right\}}\right]e^{-nt}$$

$$+2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n}{\alpha_m\left[n-\frac{k'\alpha_m^2}{R^2}\right]}\cdot\frac{J_0\left(\frac{r\alpha_m}{R}\right)}{J_1(\alpha_m)}.e^{-\frac{k'\alpha_m^2 t}{R^2}}$$

$$=T_1'+T_2'+T_3' \qquad \ldots (3.6)$$

This expression for temperature does not agree with Lal's result (1964). His expression for the temperature with the present boundary conditions does not contain T_1' and T_3' because he has solved the enrgy equation by neglecting the dissipation term and by assuming

$$T = \sum_{n=1}^{\infty} T_n(r)e^{-nt},$$

where T_n is a function of r only whereas in our case the dissipation term has not been neglected. Our expression (3.6) for the temperature is in complete agreement with similar results obtained by Ballabh (1959) and Snnedon (1951). Ballabh (1959) in his paper has obtained the expression for the velocity by using the method of super-posability and Snnedon (1951) has discussed the heat flow under exponentially decreasing temperature gradient.

Hence $T_1'+T_2'+T_3'$ is a more general solution of equation (1.1) and has been confirmed by Laplace Transform method.

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